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Intersection multiplicity of modules in the positive characteristics

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Abstract

We introduce an asymptotic length criteria on Tor_0 and Tor_1 of a pair of modules, via the Frobenius map, to study positivity and non-negativity of Serre-intersection multiplicity over local complete intersections and Gorenstein rings in the positive characteristics, when both modules have finite projective dimension. We use this criteria to derive a sufficient condition for

- (a) non-negativity over complete intersections when one of the modules has dimension ≤ 2 , and
- (b) non-negativity over Gorenstein rings of dimension ≤ 5 .

We also show that the problem of intersection multiplicity on local rings (complete intersection, Gorenstein, Cohen–Macaulay) in characteristic 0 can be reduced to local rings (complete intersection, Gorenstein, Cohen–Macaulay respectively) in the positive characteristics.

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In this note our main concern is on the question of positivity as well as non-negativity of intersection multiplicity for a pair of modules, each having finite projective dimension, over local complete intersections and local Gorenstein rings. As the title suggests, we will study this and other related questions in the positive characteristics. However, in Section 3, our main theorem will establish that similar questions in characteristic zero can be reduced to the positive characteristics. Hence, in order to prove an extended version of Serre's

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theorem on local complete intersections or Gorenstein rings in the equicharacteristic case, it will be enough to prove it in characteristic $p > 0$. For a brief introduction to the results in Sections 1 and 2, we need the following set-up.

Let R be a local ring with maximal ideal m and residue field $k = R/m$.

We assume that k is perfect and R has positive characteristic p . Let us denote by $f: R \rightarrow R$ the Frobenius map, $f(x) = x^p$, for all $x \in R$. We denote by fR the bi-algebra R having the structure of an R -algebra from the left by f^n (f repeated n times) and from the right by identity, i.e., if $\alpha \in R$, $x \in {}^fR$ then $\alpha \cdot x = \alpha^{p^n}x$ and $x \cdot \alpha = x\alpha$. By a free complex F_\bullet , we mean a complex $F_\bullet = (F_n)_{n \geq 0}$ of finitely generated free modules and we write $F^n(F_\bullet) = F_\bullet \otimes_R {}^fR$. For any module M , we write $F^n(M)$ for $M \otimes_R {}^fR$ and for any finitely generated module N write \tilde{N} for $\text{Hom}_R(H_m^r(N), E)$ where E is the injective hull of k and $r = \dim N$ (henceforth $\dim N$). Let d denote $\dim R$.

When projective dimension of M over R (henceforth $\text{pd}_R M$) is finite (or $\text{pd}_R N < \infty$), Serre's conjecture on intersection multiplicity can be generalized as follows (here we assume $\ell(M \otimes_R N) < \infty$).

Vanishing Part. $\chi(M, N) = 0$ whenever $\dim M + \dim N < \dim R$.

Positivity Part. $\chi(M, N) > 0$ whenever $\dim M + \dim N = \dim R$.

Non-negativity Part. $\chi(M, N) \geq 0$ whenever $\dim M + \dim N \leq \dim R$.

We introduced the notion of “ χ_∞ ” in [5] and obtained the following (here we are collecting several results from [5]).

Result 1. Let R be a local ring in characteristic $p > 0$. Let M and N be two finitely generated modules such that $\ell(M \otimes_R N) < \infty$ and $\text{pd}_R M < \infty$. We define

$$\chi_\infty(M, N) = \lim_{n \rightarrow \infty} \chi(F^n(M), N) / p^{n \cdot \text{codim } M}$$

and

$$\alpha_\infty(M, N) = \lim_{n \rightarrow \infty} \chi(F^n(M), N) / p^{n \cdot \dim N}$$

(when $\dim M + \dim N = \dim R$, $\chi_\infty = \alpha_\infty$) we have the following:

- (i) $\chi_\infty(M, N) = 0$ if $\dim M + \dim N < \dim R$,
- (ii) $\chi_\infty(M, N) > 0$, if M is Cohen–Macaulay and $\dim M + \dim N = \dim R$, and
- (iii) $\alpha_\infty(M, N)$ is a rational number.

When M is not Cohen–Macaulay (ii) fails to hold even in low-dimensions—this was demonstrated by the counterexample in [11]. Thus the positivity of $\chi_\infty(M, N)$, when both M and N are of finite projective dimension, M is not Cohen–Macaulay and $\dim M + \dim N = \dim R$, has been and still is one of the central issues in intersection multiplicity. Its importance has been bolstered by the fact that over complete intersections

$\chi_\infty(M, N) = \chi(M, N)$ (Corollary, Theorem 1.2). So the positivity of χ_∞ settles the positivity conjecture over complete intersections.

Unless otherwise mentioned, we assume that both modules M and N have finite projective dimension over R .

Our main theorem in Section 1 is the following.

1.2. Theorem.

- (a) Let R be Cohen–Macaulay. Then (1) the vanishing holds for χ and (2) the positivity part is valid for χ when one of the intersecting modules is Cohen–Macaulay, if for every pair of perfect modules (Cohen–Macaulay and of finite projective dimension) Q, T such that $\ell(Q \otimes T) < \infty$ and $\dim Q + \dim T = \dim R$ we have

$$\ell(F^n(Q) \otimes_R T) = p^{n \dim T} \ell(Q \otimes_R T). \quad (*)$$

- (b) If R is a complete intersection, $(*)$ holds.

As a corollary we deduce that on complete intersections $\chi_\infty(M, N) = \chi(M, N)$.

In Section 2, our main focus is on positivity as well as non-negativity for both χ and χ_∞ . First, we prove the following theorem.

2.1. Theorem. In order to study positivity and non-negativity for both χ and χ_∞ over Gorenstein rings it is enough to assume $\text{depth } M = \dim M - 1$ and $\text{depth } N = \dim N - 1$.

Our next theorem gives a criteria for positivity as well as non-negativity of χ_∞ on Gorenstein rings.

2.2. Theorem. With M, N and R as above, $\chi_\infty(M, N) > 0$ (≥ 0), if

$$\begin{aligned} & \lim_{n \rightarrow \infty} \ell(\text{Tor}_1^R(F^n(\text{Ext}_R^{s+1}(M, R)), N)) / p^{n \dim N} \\ & > (\geq) \lim_{n \rightarrow \infty} \ell(F^n(\text{Ext}_R^{s+1}(M, R)) \otimes_R N) / p^{n \dim N}, \end{aligned}$$

here $s = \dim N$.

We use the above two theorems (and a few steps in the proof of the second theorem) to prove in 2.3 (Corollary 1) that, if a specific length is zero asymptotically (see definition in 2.3), χ_∞ is non-negative over Gorenstein rings for modules M with $\dim M \leq 2$. This implies (Corollary 2 (2.3)) the non-negativity of $\chi(M, N)$ over complete intersections when $\dim M \leq 2$, provided the above “length” condition is satisfied. In Proposition 2.4 we use Corollary 1 (2.3) to derive similar results over Gorenstein rings R with $\dim R \leq 5$. In all these cases positivity will be assured provided certain limit is positive instead of being non-negative. (See last step of proof of Corollary 1.) I strongly suspect that this limit is positive; however I still do not have a proof.

In Section 3, we demonstrate that the intersection multiplicity problem over a local ring in characteristic 0 can be reduced to the positive characteristics. Our main tools are Artin's Approximation Theorem [1] and the technique of reduction introduced by Peskine and Szpiro [17]. We use the method introduced in [9] for reducing the problem from complete smooth case to the finitely generated (over a field or a discrete valuation ring) smooth case and refer to several steps in the proof of Theorem 3.6 in [9] for our proof of Theorem 3.4 here.

A brief history

In *Algèbre Locale* [20], Serre conjectured that both vanishing and positivity part are valid over regular local rings. He proved the conjecture [20] for equicharacteristic and unramified regular rings. Later Roberts [18,19] and Gillet and Soulé [14] independently proved the vanishing part; Roberts' proof covers the above case over complete intersections when both modules have finite projective dimension. In the mid-nineties Gabber [3] proved the non-negativity part over regular local rings. The non-validity of the generalized Serre's conjecture over non-smooth hypersurfaces was first demonstrated by a counter-example constructed in [11] by Hochster, McLaughlin and this author. The vanishing part over Gorenstein rings R with $\dim R \leq 5$ was also demonstrated by this author in [8]. Foxby established positivity over a local ring when grade of $M = 1$ (here N may not be of finite projective dimension!) [12]. This author proved the following special case of positivity over regular local rings [7]: p is a non-zero divisor on M , M is Cohen–Macaulay and $p^l N = 0$. Later it was extended by Kurano and Roberts [15].

We will need the following result in our work. For a proof we refer the reader to Theorem 1.5 in [6].

Result 2. *Let R be a local ring in characteristic $p > 0$ and suppose that dimension of R is d . Let F_\bullet be a complex of finitely generated free modules with finite length homology modules. Let N be a finitely generated R -module. Let W_{jn} denote the j th homology of $\text{Hom}_R(F^n(F_\bullet), N)$. We have the following.*

- (i) *If $\dim N < \dim R$, then $\lim_{n \rightarrow \infty} \ell(W_{jn})/p^{nd} = 0$.*
- (ii) *If $\dim N = \dim R$, and*
 - (a) *$j < d$, then $\lim_{n \rightarrow \infty} \ell(W_{jn})/p^{nd} = 0$;*
 - (b) *$j = d$, then $\lim_{n \rightarrow \infty} \ell(W_{dn})/p^{nd} = \lim_{n \rightarrow \infty} \ell(F^n(H_0(F_\bullet)) \otimes \tilde{N})/p^{nd}$ (which is positive), and*
 - (c) *$j > d$, then $\lim_{n \rightarrow \infty} \ell(W_{jn})/p^{nd} = \lim_{n \rightarrow \infty} \ell(H_{j-d}(F^n(F_\bullet)) \otimes \tilde{N})/p^{nd}$.*

Throughout this work all rings are local and all modules are finitely generated unless otherwise mentioned. We concentrate mainly on the situation where both modules M and N have finite projective dimension and $\ell(M \otimes_R N) < \infty$. In Sections 1 and 2 the local ring R has characteristic $p > 0$. However, Lemma 1.1 and Theorem 2.1 are valid in any characteristic.

1. Section

1.1. Lemma. *Let R be Cohen–Macaulay. For the vanishing part to be valid over R it is enough to consider the case when $\text{pd}_R M < \infty$, $\ell(M \otimes_R N) < \infty$, M and N are Cohen–Macaulay and $\dim M + \dim N = \dim R - 1$.*

Proof. For the vanishing to hold we need to show that $\chi(M', N') = 0$ for any pair of modules M', N' such that $\ell(M' \otimes_R N') < \infty$, $\dim M' + \dim N' < \dim R$ and $\text{pd}_R M' < \infty$. Since R is Cohen–Macaulay, the above conditions imply that $\text{ht ann}_R M' + \text{ht ann}_R N' > \dim R$ and $\text{ann}_R M' + \text{ann}_R N'$ is m -primary where m is the maximal ideal of R . Let $\{x_1, \dots, x_t\}$ be a maximal R -sequence contained in $\text{ann}_R N'$ such that $\ell(M'/\underline{x}M') < \infty$. Here \underline{x} denotes the ideal generated by $\{x_1, \dots, x_t\}$. Any short exact sequence

$$0 \rightarrow N_1 \rightarrow (R/\underline{x})^{\alpha_0} \rightarrow N' \rightarrow 0$$

implies that $\chi(M', N') = -\chi(M', N_1)$, $\text{depth } N_1 = \text{depth } N' + 1$ and $\dim N_1 = \dim N'$. Thus, after a finite number of steps, we obtain modules M_s, N_r such that M_s, N_r are Cohen–Macaulay and $\dim M_s = \dim M'$, $\dim N_r = \dim N'$ and $\chi(M_s, N_r) = \chi(M', N')$. Since $t > \dim M'$, we consider R -sequences $\{x_1, \dots, x_i\}$ contained in $\text{ann } N_r$, such that $\ell(M_s/(x_1, \dots, x_i)M_s) < \infty$ and $\dim M' + 1 \leq i \leq t$. Now consider a short exact sequence of the type:

$$0 \rightarrow N_{r+1} \rightarrow \bigoplus_1^{a_r} R/(x_1, \dots, x_{t-1}) \rightarrow N_r \rightarrow 0.$$

Then $\chi(M_s, N_r) = -\chi(M_s, N_{r+1})$, N_{r+1} is Cohen–Macaulay and $\dim N_{r+1} = \dim N_r + 1$. Applying this method a finite number of times, we obtain our required M, N . \square

1.2. Theorem.

(a) *Let R be Cohen–Macaulay. Then (1) the vanishing holds for χ and (2) the positivity is valid for χ when one of the intersecting modules is Cohen–Macaulay, if for any pair of perfect modules Q, T such that $\ell(Q \otimes_R T) < \infty$ and $\dim Q + \dim T = \dim R$, we have*

$$\ell(F^n(Q) \otimes_R T) = p^{n \dim T} \ell(Q \otimes_R T). \quad (*)$$

(b) *(*) in part (a) is valid over local complete intersections.*

Proof. (a) *Vanishing.* By Lemma 1.1, it is enough to prove the following: Let M and N be two perfect modules such that $\ell(M \otimes_R N) < \infty$, and $\dim M + \dim N = \dim R - 1$; then $\chi(M, N) = 0$. Let $\{x_1, \dots, x_t\}$ be an R -sequence contained in $\text{ann}_R N$ such that $t = \dim M$ and $\ell(M/\underline{x}M) < \infty$. Consider the exact sequence

$$0 \rightarrow T \rightarrow (R/\underline{x})^r \rightarrow N \rightarrow 0.$$

Then T is perfect, $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim T = \dim R$. Let $s = \dim T = \dim R/\underline{x}$. By hypothesis $\ell(F^n(M) \otimes_R T) = p^{ns} \ell(M \otimes_R T)$ and $\ell(F^n(M) \otimes_R (R/\underline{x})^r) = p^{ns} \ell(M \otimes_R (R/\underline{x})^r)$. Hence $\chi(F^n(M), N) = p^{ns} \chi(M, N)$.

Thus $\chi_\infty(M, N) = \chi(M, N)$. On the other hand, by Result 1, we have $\chi_\infty(M, N) = 0$. Hence $\chi(M, N) = 0$.

Positivity. Let M be perfect. Let N be an R -module such that $\ell(M \otimes_R N) < \infty$, $\text{pd}_R N < \infty$, and $\dim M + \dim N = \dim R$. By considering a free resolution of N over R/\underline{x} truncated at the right spot

$$0 \rightarrow T \rightarrow (R/\underline{x})^{c_r} \rightarrow \cdots \rightarrow (R/\underline{x})^{c_0} \rightarrow N \rightarrow 0$$

where \underline{x} denotes the ideal generated by a maximal R -sequence contained in $\text{ann}_R N$ such that $\ell(M/\underline{x}M) < \infty$ and T is perfect, we obtain (by hypothesis)

$$\chi(F^n(M), N) = p^{ns} \chi(M, N), \quad \text{where } s = \dim N = \text{codim } M.$$

Thus $\chi_\infty(M, N) = \chi(M, N)$. On the other hand by Proposition 1.2 [6] or by Result 1 we have $\chi_\infty(M, N) > 0$. Thus $\chi(M, N) > 0$.

(b) For this part we use local Chern characters, the Riemann–Roch theorem (due to Fulton and MacPherson) and the compatibility of local Chern characters with the Frobenius functor [19]. Let Q and T be two perfect modules such that $\ell(Q \otimes T) < \infty$, $\dim Q = r$, $\dim T = s$ and $\dim Q + \dim T = \dim R$. Let F_\bullet be a minimal free resolution of Q and G_\bullet be a minimal free resolution of T . Then $F^n(F_\bullet)$ is a minimal free resolution of $F^n(Q)$. Note that $\text{Tor}_i(F^n(Q), T) = 0$ for $i > 0$.

Hence

$$\begin{aligned} \ell(F^n(Q) \otimes_R T) &= \chi(F^n(F_\bullet) \otimes_R G_\bullet) = \text{ch}(F^n(F_\bullet)) \text{ch}(G_\bullet)(\tau(A)) \\ &= \sum \text{ch}_i(F^n(F_\bullet)) \text{ch}_{d-i}(G_\bullet)[A] = \text{ch}_s F^n(F_\bullet) \text{ch}_r(G_\bullet)[A] \\ &= p^{ns} \text{ch}_s(F_\bullet) \text{ch}_r(G_\bullet)[A] = p^{ns} \chi(F_\bullet \otimes G_\bullet) = p^{ns} \ell(Q \otimes_R T). \end{aligned}$$

(For notations, see [13,19].) \square

Corollary. Let R be local complete intersection. Let M, N be two finitely generated R -modules such that $\text{pd}_R M < \infty$, $\text{pd}_R N < \infty$, $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N \leq \dim R$. Then $\chi_\infty(M, N) = \chi(M, N)$.

Proof. When $\dim M + \dim N < \dim R$, $\chi_\infty(M, N) = 0 = \chi(M, N)$. So, we can assume $\dim M + \dim N = \dim R$. Let $r = \text{height ann}_R M = \dim N$. Choose $\{x_1, \dots, x_r\}$, an R -sequence contained in $\text{ann}_R M$, such that $\ell(N/\underline{x}N) < \infty$. Here \underline{x} denotes the ideal (x_1, \dots, x_r) . Write $\bar{R} = R/\underline{x}$. Consider an exact sequence

$$0 \rightarrow T \rightarrow \bar{R}^{t_s} \rightarrow \bar{R}^{t_{s-1}} \rightarrow \cdots \rightarrow \bar{R}^{t_0} \rightarrow M \rightarrow 0$$

where T is perfect. Now applying F^n to the above complex and tensoring it with N we obtain our required result from the above theorem. \square

One can also give an independent proof of the result in the corollary by arguing as in the proof of part (b) of the above theorem.

Remarks. (1) I believe that the length equality in Theorem 1.2 can be proved without using local Chern characters—though I am not able to produce such a proof at present!

(2) The relation (*) of part (a) does not hold good in general over Cohen–Macaulay (counterexample due to Roberts [19]) or Gorenstein rings (counterexample due to Miller and Singh [16]).

2. Section

In this section we study positivity for pairs of modules both having finite projective dimension. We first state a theorem, due to Auslander and Bridger (Theorem 4.26 [2]).

Theorem. *Let M be a finitely-generated module over a Gorenstein ring R . There exists a short exact sequence*

$$0 \rightarrow L \rightarrow M \oplus R^t \rightarrow Q \rightarrow 0,$$

where Q is a module with $\text{pd}_R Q < \infty$ and L is a maximal Cohen–Macaulay module.

We use this result to prove our next theorem.

2.1. Theorem. *Let (R, m) be a Gorenstein ring and suppose that $\text{pd}_R M < \infty$, $\text{pd}_R N < \infty$. In order to prove the positivity for $\chi(M, N)$ it is enough to assume that $\text{depth } M \geq \dim M - 1$, $\text{depth } N \geq \dim N - 1$, $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N = \dim R$.*

Proof. Let us start with any M, N . Let $\dim M = r$ and $\dim N = s$ and $I = \text{ann}_R M$. Since $\ell(M \otimes_R N) < \infty$ and $r + s = n$, it follows that I contains an R -sequence $\{x_1, \dots, x_s\}$ which forms a system of parameters for N . We write $\bar{R} = R/(x_1, \dots, x_s)$. If $\text{pd}_{\bar{R}} M < \infty$, $\text{rank}_{\bar{R}} M$ is defined to be the rank $S^{-1}M$, where S denotes the set of non-zero divisors of \bar{R} ; this rank is well defined as finite projective dimension of M forces each M_P to have the same rank over \bar{R}_P for each minimal prime ideal P of \bar{R} . From the exact sequence $0 \rightarrow \bar{R}^d \rightarrow M \rightarrow H \rightarrow 0$, where $\dim H < \dim M$ and $d = \text{rank}_{\bar{R}} M$, it follows that $\chi(M, N) = d\chi(\bar{R}, N) > 0$. So we assume $\text{pd}_{\bar{R}} M = \infty$. By the theorem stated above, we obtain an exact sequence

$$0 \rightarrow L \rightarrow \bar{R}^t \oplus M \rightarrow Q \rightarrow 0, \quad (1)$$

where Q is a module with $\text{pd}_{\bar{R}} Q < \infty$ and L is a maximal Cohen–Macaulay module over \bar{R} . If $\dim Q = \dim \bar{R}$, let $d' = \text{rank}_{\bar{R}} Q$, we obtain an exact sequence

$$0 \rightarrow \bar{R}^{d'} \rightarrow Q \rightarrow Q' \rightarrow 0, \quad (2)$$

where Q' is a module of finite projective dimension over \bar{R} and $\dim Q' < \dim \bar{R}$. Now from

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \bar{R}^{d'} & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \bar{R}^t \oplus M & \longrightarrow & Q \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & Q' & & \downarrow \\ & & & & 0 & & \end{array}$$

we obtain the following short exact sequence:

$$0 \rightarrow L \oplus \bar{R}^{d'} \rightarrow \bar{R}^t \oplus M \rightarrow Q' \rightarrow 0.$$

This shows that we can assume $\dim Q < \dim \bar{R}$ in (1) and hence $\chi(Q, N) = 0$ (by an easy spectral sequence argument). From (1) we obtain $\chi(M, N) = -\chi(\bar{R}^t, N) + \chi(L, N)$. On the other hand, since $\text{rank of } L \geq t$ we obtain the following short exact sequence:

$$0 \rightarrow \bar{R}^t \rightarrow L \rightarrow M' \rightarrow 0.$$

This gives us $\chi(M', N) = \chi(L, N) - \chi(\bar{R}^t, N)$. Thus $\chi(M, N) = \chi(M', N)$, where $\text{depth } M' \geq \dim M - 1$.

Now repeating the same process with N we obtain an N' such that $\chi(M', N') = \chi(M', N)$ and $\text{depth } N' \geq \dim N' - 1$. Thus the desired result follows. \square

Remarks. (1) The above theorem is valid even if $\text{pd}_R N = \infty$.

(2) Similar result holds for positivity of χ_∞ .

(3) If $\text{depth } M' = \dim M'$, i.e., M' is Cohen–Macaulay, we have $\chi_\infty(M', N') > 0$ by Result 1. Hence we concentrate on the case when $\text{depth } M' = \dim M' - 1$ and $\text{depth } N' = \dim N' - 1$.

2.2. Theorem. Let R be a local Gorenstein ring. Let M and N be two modules such that $\ell(M \otimes N) < \infty$, $\text{pd}_R M < \infty$, $\text{pd}_R N < \infty$, $\text{depth } M = \dim M - 1$, $\text{depth } N = \dim N - 1$ and $\dim M + \dim N = \dim R$. Then $\chi_\infty(M, N) > 0$ (≥ 0) if

$$\lim_{n \rightarrow \infty} \ell(\text{Tor}_1^R(F^n(\text{Ext}_R^{s+1}(M, R)), N))/p^{ns} > (\geq) \lim_{n \rightarrow \infty} \ell(F^n(\text{Ext}_R^{s+1}(M, R)) \otimes_R N)/p^{ns},$$

here $s = \dim N$.

Proof. Let $r = \dim M$, $s = \dim N$; then $r + s = d = \dim R$.

Since $\text{depth } M + \text{depth } N = \dim R - 2$, $\text{Tor}_i^R(F^n(M), N) = 0$ for $i > 2$.

Let

$$F_\bullet: 0 \rightarrow R^{a_{s+1}} \xrightarrow{\phi_{s+1}} R^{a_s} \xrightarrow{\phi_s} \dots \rightarrow R^{a_0} \rightarrow 0$$

be a minimal free resolution of M . Then

$$F^n(F_\bullet): 0 \rightarrow R^{a_{s+1}} \xrightarrow{\phi_{s+1}^{[p^n]}} R^{a_s} \rightarrow \dots \rightarrow R^{a_0} \rightarrow 0$$

is a minimal free resolution of $F^n(M)$ ($\phi_i^{[p^n]}$ is the matrix obtained from ϕ_i by raising its entries to p^n th power). Write $F^n(G_s) = \text{coker } \phi_s^{[p^n]*}$, where $(-)* = \text{Hom}_R(-, R)$. We obtain two short exact sequences

$$\begin{aligned} 0 \rightarrow \text{Ext}_R^s(F^n(M), R) &\rightarrow F^n(G_s) \rightarrow \text{Im } \phi_{s+1}^{[p^n]*} \rightarrow 0 \quad \text{and} \\ 0 \rightarrow \text{Im } \phi_{s+1}^{[p^n]*} &\rightarrow R^{a_{s+1}} \rightarrow F^n(\text{Ext}_R^{s+1}(M, R)) \rightarrow 0. \end{aligned} \quad (1)$$

By Result 2, we have the following

$$\begin{aligned} \lim_{n \rightarrow \infty} \ell(\text{Ext}_R^s(F^n(M), N))/p^{ns} &= \lim_{n \rightarrow \infty} \ell(F^n(M) \otimes_R N)/p^{ns}, \\ \lim_{n \rightarrow \infty} \ell(F^n(\text{Ext}_R^{s+1}(M, R)) \otimes_R N)/p^{ns} &= \lim_{n \rightarrow \infty} \ell(\text{Tor}_1^R(F^n(M), N))/p^{ns}, \end{aligned} \quad (2)$$

and

$$0 = \lim_{n \rightarrow \infty} \ell(\text{Tor}_2^R(F^n(M), N))/p^{ns}.$$

(To see this, let $I = \text{ann}_R N$. Write $\bar{R} = R/I$. Then $F^n(F_\bullet) \otimes N = F_{\bar{R}}^n(\bar{F}_\bullet) \otimes_R N$ where $\bar{F}_\bullet = F_\bullet \otimes \bar{R}$. Note that $\ell(H_i(\bar{F}_\bullet)) < \infty$ for $i \geq 0$. Now we apply Result 2.)

Also note that, by Result 2, we have:

$$\lim_{n \rightarrow \infty} \ell(\text{Tor}_i^R(F^n(G_s), N))/p^{ns} = \lim_{n \rightarrow \infty} \text{Ext}_R^{s-i}(F^n(M), N)/p^{ns} = 0 \quad \text{for } i > 0. \quad (3)$$

Now, from the short exact sequences in (1), we obtain the following exact sequence

$$\begin{aligned} &\rightarrow \operatorname{Tor}_2^R(F^n(\operatorname{Ext}_R^{s+1}(M, R)), N) \xrightarrow{\alpha_n} \operatorname{Ext}_R^s(F^n(M), R) \otimes N \\ &\xrightarrow{\beta_n} \operatorname{Ext}_R^s(F^n(M), N) \xrightarrow{\gamma_n} \operatorname{Tor}_1^R(F^n(\operatorname{Ext}_R^{s+1}(M, R)), N) \rightarrow 0. \end{aligned} \quad (4)$$

Note that $\lim_{n \rightarrow \infty} \ell(\operatorname{Ker} \alpha_n) / p^{nd} = 0$ by (3). Hence by (2), (3) and (4) we have

$$\begin{aligned} \chi_\infty(M, N) &= \lim_{n \rightarrow \infty} \frac{1}{p^{ns}} [\ell(\operatorname{Ext}_R^s(F^n(M), N)) - \ell(\operatorname{Ext}_R^{s+1}(F^n(M), N))] \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{p^{ns}} [\ell(\operatorname{Tor}_1^R(F^n(\operatorname{Ext}_R^{s+1}(M, R)), N)) \\ &\quad - \ell(F^n(\operatorname{Ext}_R^{s+1}(M, R)) \otimes_R N)]. \end{aligned}$$

Thus the theorem is proved. \square

Remark. The criteria mentioned above for positivity fails to hold when $\operatorname{pd}_R N = \infty$. For this we refer the reader to the counterexample in [11].

2.3. A definition and corollaries to the above theorem.

Definition. Let R be a local Gorenstein ring. We say that R satisfies the “length” condition if the following holds: For any pair of finitely generated R modules M, N of finite projective dimension such that $\ell(M \otimes_R N) < \infty$, $\dim M = 2$, $\dim N = s$, $s + 2 = \dim R$, $\operatorname{depth} M = \dim M - 1$ and $\operatorname{depth} N = \dim N - 1$,

$$\lim_{n \rightarrow \infty} \ell(\operatorname{Ext}^3(N, R) \otimes H_m^0(F^n(\operatorname{Ext}^{s+1}(M, R)))^\vee) / p^{ns} = 0.$$

For any module T , $T^\vee = \operatorname{Hom}_R(T, E)$, where E is the injective hull of the residue field.

Corollary 1. Let R be a Gorenstein local ring and suppose that R satisfies the “length” condition. Let M, N be two modules of finite projective dimension such that $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N = \dim R$. Suppose that $\dim M \leq 2$. Then $\chi_\infty(M, N) \geq 0$.

Proof. By Theorem 2.1 we can assume $\operatorname{depth} M = \dim M - 1$ and $\operatorname{depth} N = \dim N - 1$. Write $s = \dim N$. Since $\dim M \leq 2$, $\dim \operatorname{Ext}^{s+1}(M, R) \leq 1$. The same is true for $\dim F^n(\operatorname{Ext}^{s+1}(M, R)) (= \operatorname{Ext}^{s+1}(F^n(M), R))$. Hence $\chi(N, F^n(\operatorname{Ext}^{s+1}(M, R))) = 0$ (Proposition 1.3 [4]). Since $\operatorname{depth} N = s - 1 = \dim R - \dim M - 1 \geq \dim R - 3$, we can assume $\operatorname{depth} N = \dim R - 3$ (for $\operatorname{depth} N = \dim R - 2 = \dim N \Rightarrow N$ is Cohen–Macaulay and this case is easily verifiable). Thus we have

$$\chi(F^n(\operatorname{Ext}_R^{s+1}(M, R)), N) = \sum_{i=0}^3 \ell(\operatorname{Tor}_i^R(F^n(\operatorname{Ext}_R^{s+1}(M, R)), N)) = 0.$$

However,

$$\begin{aligned}
& \text{Ext}^d(\text{Tor}_3^R(F^n(\text{Ext}_R^{s+1}(M, R)), N), R) \\
& \simeq \text{Ext}^3(N, \text{Ext}^d(F^n(\text{Ext}_R^{s+1}(M, R)), R)) \\
& \simeq \text{Ext}_R^3(N, R) \otimes_R \text{Ext}^d(F^n(\text{Ext}_R^{s+1}(M, R)), R).
\end{aligned}$$

Since $\dim N = s$ and $\text{depth } N = s - 1 = \dim R - 3$, we have $\text{pd}_R N = 3$ and $\dim \text{Ext}^3(N, R) < s$. Hence, by our assumption, $\lim_{n \rightarrow \infty} \ell(\text{Tor}_3^R(F^n(\text{Ext}_R^{s+1}(M, R)), N))/p^{ns} = 0$.

Thus, from the above equation, we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{p^{ns}} [\ell(F^n(\text{Ext}_R^{s+1}(M, R)) \otimes_R N) - \ell(\text{Tor}_1^R(F^n(\text{Ext}_R^{s+1}(M, R)), N)) \\
& \quad + \ell(\text{Tor}_2^R(F^n(\text{Ext}_R^{s+1}(M, R)), N))] = 0.
\end{aligned} \tag{5}$$

Hence from (2), (3), (4) in the proof of Theorem 2.2 and from (5) above we obtain

$$\begin{aligned}
\chi_\infty(M, N) &= \lim_{n \rightarrow \infty} \frac{1}{p^{ns}} [\ell(\text{Ext}_R^s(F^n(M), N)) - \ell(F^n(\text{Ext}_R^{s+1}(M, R)) \otimes_R N)] \\
&= \lim_{n \rightarrow \infty} \frac{1}{p^{ns}} [\ell(\text{Tor}_1^R(F^n(\text{Ext}_R^{s+1}(M, R)), N)) + \ell(\text{Ext}_R^s(F^n(M), R) \otimes_R N) \\
&\quad - \ell(\text{Tor}_2^R(F^n(\text{Ext}_R^{s+1}(M, R)), N)) - \ell(F^n(\text{Ext}_R^{s+1}(M, R)) \otimes_R N)] \\
&= \lim_{n \rightarrow \infty} \frac{1}{p^{ns}} [\ell(\text{Ext}_R^s(F^n(M), R) \otimes_R N)] \\
&\geq 0. \quad \square
\end{aligned}$$

Remark. I strongly feel that the above limit is positive, but I still do not have a proof!

Corollary 2. Let R be a complete intersection and let M, N be as in Corollary 1. Suppose that R satisfies the “length” condition. Then $\chi(M, N) \geq 0$.

Proof follows easily from the above corollary and the fact that on complete intersection $\chi_\infty(M, N) = \chi(M, N)$ for the above set-up (Corollary to Theorem 1.2, Section 1).

2.4. Proposition. Let R be Gorenstein with $\dim R \leq 5$. Then vanishing holds. Non-negativity will also hold provided R satisfies the “length” condition.

Proof. For vanishing, we refer the reader to Theorem 1 in [8].

For positivity of $\chi(M, N)$, can assume $\text{depth } M = \dim M - 1$, $\text{depth } N = \dim N - 1$, $\dim M + \dim N = 5$ (Theorem 2.1).

If $\dim M = 1$, $\dim N = 4$ and we are done by Foxby’s theorem.

So the main case to be studied is when $\dim M = 2$, $\dim N = 3$. Then

$$\chi(F^n(M), N) = \sum_{\substack{\dim R/P = \dim M = 2 \\ P \in \text{Ass}_R(M)}} \ell(F^n(M)_P) \chi(R/P, N)$$

(since $\chi(R/\mathfrak{q}, N) = 0$ if $\dim R/\mathfrak{q} \leq 1$ by Proposition 1.3 in [4]). By Proposition 1.14 in [5] we have $\ell(F^n(M)_P) = p^{3n} \ell(M_P)$ (since $\text{ht } P = 3$, R_P is a 3-dimensional Gorenstein ring and M_P is a module of finite length and finite projective dimension over R_P). Thus

$$\chi(F^n(M), N) = p^{3n} \sum \ell(M_P) \chi(R/P, N) = p^{3n} \chi(M, N).$$

Thus $\chi_\infty(M, N) = \chi(M, N)$.

Now we are done by Corollary 1. \square

3. Section

The main theorem, we want to prove in this section, is the following:

3.1. Theorem. *Let B be a local ring of characteristic 0 and let M, N be two finitely generated B -modules such that*

- (i) $\text{pd}_B M < \infty$, and $\text{pd}_B N < \infty$,
- (ii) $\ell(M \otimes_B N) < \infty$, and
- (iii) $\dim M + \dim N \leq \dim B$.

Then there exists a local ring B' of characteristic $p > 0$ and two finitely generated B' -modules M', N' of finite projective dimension such that $\ell(M' \otimes_{B'} N') < \infty$, $\dim M' + \dim N' \leq \dim B'$ and $\chi^{B'}(M', N')$ has the same sign as $\chi^B(M, N)$. Moreover if B is local complete intersection (Gorenstein, Cohen–Macaulay, etc.), B' can also be chosen to be the same.

We will prove this theorem in two steps. The carry-over of the structure of B , i.e., Gorenstein, etc. to B' will be clear from our proof.

Step 1.

3.2. Proposition. *Let A be a local ring of characteristic 0, essentially of finite type over a field k . Let M, N be two finitely generated A -modules of finite projective dimension such that $\ell(M \otimes_A N) < \infty$ and $\dim M + \dim N \leq \dim A$. Then there exist a local ring A' of characteristic $p > 0$, essentially of finite type over a finite field and a pair of finitely generated A' -modules M', N' of finite projective dimension such that $\ell(M' \otimes_{A'} N') < \infty$, $\dim M' + \dim N' \leq \dim A'$ and $\chi^{A'}(M', N')$ has the same sign as $\chi^A(M, N)$.*

Proof. The proof of this theorem follows immediately from Peskine–Szpiro’s work in [17]. Arguing exactly as in Lemmas 2.2, 2.3 and 2.4 in the proof of Theorem 2.1 in the above paper we obtain our required result. We leave the details to the reader. \square

Step 2. Here we would like to reduce intersection multiplicity from equicharacteristic complete local case to the localized affine-algebra case. Our main tool here is Artin’s Approxi-

mation Theorem [1]. Let (S, m) denote a complete local ring in the equicharacteristic case, i.e., $S = \hat{A}/\hat{J}$, where $\hat{A} = k[[x_1, \dots, x_d]]$, a power series ring in d variables, k —a field and \hat{J} is an ideal of \hat{A} , say $\hat{J} = (f_1, \dots, f_\ell)$. Write $A = k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$, A^h = the henselization of A . Recall that $A^h = \lim_{n \in \beta} A_n$, where each A_n is an étale neighbourhood of A . We know, by Artin's Approximation Theorem, that A^h is an approximation ring, i.e. given any system of equations $F_1 = 0, \dots, F_r = 0$, $F_i \in A[Y_1, \dots, Y_u]$, if the system has a solution $\hat{x}_1, \dots, \hat{x}_u$ in \hat{A} , then for every $t > 0$, the system has a solution $x_1^{(t)}, \dots, x_u^{(t)}$ in A^h such that $x_i^{(t)} \equiv \hat{x}_i \pmod{\hat{m}^t}$, $\forall i, 1 \leq i \leq u$. Since, for any such system of equations, any solution consists of finite number of elements, we can always assume that these $x_i^{(t)}$ s lie in A_n for some $n \in \beta$. We will use this fact to reduce our problem to a similar problem on a local ring R , essentially finite type over k . We achieve this by closely following the steps worked out in Section 3 of [9].

We start with a theorem and a corollary, both due to Peskine and Szpiro.

3.3. Theorem [17, 6.2]. *Let V be a field or an excellent d.v.r., and let A be a local ring of essentially finite type over V . Let \hat{A} denote the completion of A with respect to the maximal ideal m of A . Let M be a finitely generated module of finite projective dimension. Consider a minimal free resolution of M over \hat{A} :*

$$0 \rightarrow L_r \rightarrow L_{r-1} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0.$$

Then, for every integer c , there exists an exact complex:

$$0 \rightarrow P_r \rightarrow P_{r-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0,$$

of finitely generated free modules over A^h such that $P_\bullet \otimes_{A^h} (A^h/m^c A^h) = L_\bullet \otimes_{\hat{A}} (\hat{A}/m^c \hat{A})$.

Corollary [17, 6.3]. *Let A, M be as above. Then for every integer c , there exists an $n \in I$ and an A_n -module M_n of finite projective dimension such that*

- (i) $M_n/m^c M_n \simeq M/m^c M$.
- (ii) $\text{pd}_{A_n} M_n = \text{pd}_{\hat{A}} M$.

For proof, we refer the reader to [17]. Note that the complex P_\bullet in the theorem, can also be considered as a complex over A_n , for some $n \in I$.

We now return to our notations described in the beginning of this section.

Notation. Given $t > 0$, let $g_1, \dots, g_\ell \in A_n$, for some $n \in \beta$, be such that $g_i \equiv f_i \pmod{\hat{m}^t}$. We denote by R_n the local ring $A_n/(g_1, \dots, g_\ell)$, and by m_n the maximal ideal of R_n .

In such a case $R_n/m_n^t \simeq S/m^t$. Note that R_n is not necessarily unique and it is of essentially finite type over k . Moreover, by Theorem 3.3, we have $\text{depth } R_n = \text{depth } S$ and $\text{Tor}_i^{A_n}(R_n, A_n/m_n^t) \simeq \text{Tor}_i^{\hat{A}}(S, \hat{A}/\hat{m}^t)$ for $i \geq 0$.

3.4. Lemma. Let M be a finitely generated module over S . Then, given $t > 0$, $\exists n \in \beta$, a local ring R_n and a finitely generated module M_n over R_n such that $M_n/m_n^t M_n \simeq M/m^t M$.

Proof. Since M is an \hat{A} module, $\exists M_n$ over A_n such that $M_n/m_{A_n}^t M_n = M/m^t M$ (Corollary, Theorem 3.3). Here m_{A_n} denotes the maximal ideal of A_n . We need to assure that M_n is an R_n -module. For that, it is enough to approximate the equations involved in the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} \bigoplus_1^r \hat{A}^\ell & \xrightarrow{\psi} & \hat{A}^r & \longrightarrow & S^r & \longrightarrow & 0 \\ \downarrow & & \parallel \text{id} & & \downarrow & & \\ \hat{A}^s & \longrightarrow & \hat{A}^r & \longrightarrow & M & \longrightarrow & 0 \end{array} \quad (*)$$

where $\hat{A}^\ell \rightarrow \hat{A}$ is defined by $e_i \rightarrow f_i$. \square

Remark. By our construction, $\text{Tor}_i^{A_n}(M_n, A_n/m_{A_n}^t) \simeq \text{Tor}_i^{\hat{A}}(M, \hat{A}/\hat{m}^t)$.

3.5. Proposition. Let $0 \rightarrow N \rightarrow M \rightarrow T \rightarrow 0$ be an exact sequence of finitely generated S -modules. Then, for every integer t , there exists an $n \in \beta$ and an exact sequence of finitely generated R_n -modules

$$0 \rightarrow N_n \rightarrow M_n \rightarrow T_n \rightarrow 0$$

such that

- (i) $N_n/m_n^t N_n \simeq N/m^t N$, $M_n/m_n^t M_n \simeq M/m^t M$ and $T_n/m_n^t T_n \simeq T/m^t T$. Moreover, $\text{Tor}_i^{A_n}(N_n, A_n/m_{A_n}^t) \approx \text{Tor}_i^{\hat{A}}(N, \hat{A}/\hat{m}^t)$ for every $i \geq 0$; similar isomorphisms hold for M , M_n and T , T_n respectively.
- (ii) If M is S -free of rank r , so is M_n over R_n .

Proof. The proof of Proposition 3.3 in [9] establishes the existence of N_n , M_n , T_n as described above over A_n . In addition, imposing the solutions of equations in (*) of the above lemma for each of N , M and T , we are done. \square

Corollary 1. Given a finite complex L_\bullet of finitely generated modules over S and a positive integer t , we can find $n \in \beta$ and a finite complex $L_{\bullet,n}$ of finitely generated modules over a local ring R_n such that

- (i) $L_\bullet \otimes S/m^t \simeq L_{\bullet,n} \otimes R_n/m_n^t$,
- (ii) $H_i(L_\bullet) \otimes S/m^t \simeq H_i(L_{\bullet,n}) \otimes R_n/m_n^t$, and
- (iii) when L_\bullet is a free-complex, so is $L_{\bullet,n}$.

For a proof, break up L_\bullet into short exact sequences and apply the above proposition.

Corollary 2. *Given an integer t , a finitely generated module M with $\text{pd}_S M < \infty$ and F_\bullet a finite free resolution of M over S , there exists an $n \in \beta$, a local ring R_n , a finitely generated module M_n over R_n , and a finite free resolution $F_{\bullet n}$ of M_n over R_n such that*

- (i) $M_n/m_n^t M_n \simeq M/m^t M$, and
- (ii) $F_{\bullet n} \otimes R_n/m_n^t \simeq F_\bullet \otimes S/m^t$.

The proof follows easily from Corollary 1 above.

Remarks. (1) Let M be a finitely generated S -module. Let F_\bullet be a minimal free resolution of M over S . Given any integer $t > 0$, $n \in \beta$, a local ring R_n and a finitely generated R_n -module M_n , in order that a minimal free resolution $F_{\bullet n}$ of M_n exists such that $F_{\bullet n} \otimes R_n/m_n^t \simeq F_\bullet \otimes S/m^t$, it is both necessary and sufficient that $\text{Tor}_i^{R_n}(M_n, R_n/m_n^t) \simeq \text{Tor}_i^S(M, S/m^t)$, $\forall i \geq 0$ functorially.

(2) When $\text{pd}_S M < \infty$, the above remark implies that $\text{pd}_{R_n} M_n < \infty$, provided the above Tor condition is satisfied; moreover in such a case $\text{depth}_{R_n} M_n = \text{depth}_S M$.

3.6. Theorem. *Let M, N be two finitely generated S -modules such that $\text{pd}_S M < \infty$, $\text{pd}_S N < \infty$ and $\ell(M \otimes_S N) < \infty$. Then, for every integer t , there exists $n \in \beta$, a local ring R_n , two finitely generated R_n -modules M_n, N_n such that*

- (i) $\text{pd}_{R_n} M_n < \infty$, $\text{pd}_{R_n} N_n < \infty$,
- (ii) $\ell(M_n \otimes_{R_n} N_n) < \infty$, and
- (iii) $\text{Tor}_i^{R_n}(M_n, N_n) \otimes_{R_n} R_n/m_n^t \simeq \text{Tor}_i^S(M, N) \otimes S/m^t$, $\forall i \geq 0$.

Proof. Existence of M_n, N_n and (i) follows from Corollary 3.3. (ii) follows from similar arguments as in Lemma 3.2.1 in [9]. Let F_\bullet, G_\bullet be minimal free resolutions of M and N respectively over S and let $F_{\bullet n}, G_{\bullet n}$ be corresponding minimal free resolutions of M_n and N_n respectively over R_n . Then $F_{\bullet n} \otimes_{R_n} R_n/m_n^t \simeq F_\bullet \otimes_S S/m^t$, $G_{\bullet n} \otimes_{R_n} R_n/m_n^t \simeq G_\bullet \otimes_S S/m^t$, and $(F_{\bullet n} \otimes_{R_n} G_{\bullet n}) \otimes_{R_n} R_n/m_n^t \simeq (F_\bullet \otimes_S G_\bullet) \otimes_S S/m^t$. Since $\ell(M \otimes_S N) < \infty$ ($\ell(M_n \otimes_{R_n} N_n) < \infty$), we have $\ell(\text{Tor}_i^S(M, N)) < \infty$ ($\ell(\text{Tor}_i^{R_n}(M_n, N_n)) < \infty$) $\forall i \geq 0$. Let $r = \text{depth } S$. Then, by construction, $r = \text{depth } R_n$, hence the possible highest value of i such that $\text{Tor}_i^S(M, N)(\text{Tor}_i^{R_n}(M_n, N_n)) \neq 0$ is r . Given $F_\bullet \otimes_S G_\bullet$, we can construct a finite complex L_\bullet of finitely generated S -modules

$$L_\bullet: 0 \rightarrow T_{2r} \rightarrow L_{2r-1} \rightarrow \cdots \rightarrow L_{r-1} \rightarrow \cdots \rightarrow L_0 \rightarrow 0$$

such that L_i s are all free except T_{2r} possibly, a map $\phi_\bullet: L_\bullet \rightarrow (F_\bullet \otimes_S G_\bullet) \geq 1$ such that the mapping cone P_\bullet of ϕ_\bullet is exact—except at $i = 0$ and $H_0(P_\bullet) = M \otimes_S N$. We have the following diagram:

$$\begin{array}{ccccccc}
0 \rhd T_{2r} & \longrightarrow & L_{2r-1} & \longrightarrow & L_{2r-2} & \longrightarrow & \cdots \longrightarrow L_0 \longrightarrow 0 \\
& & \downarrow \phi_{2r-1} & & \downarrow \phi_{2r-2} & & \downarrow \phi_0 \\
0 \rhd (F_\bullet \otimes_S G_\bullet)_{2r} & \rhd & (F_\bullet \otimes_S G_\bullet)_{r-1} & \rhd & \cdots \rhd & (F_\bullet \otimes_S G_\bullet)_1 & \rhd (F_\bullet \otimes_S G_\bullet)_0 \rhd 0
\end{array}$$

We denote the above diagram by (**) (see Proposition 1.1 in [10]). \square

Given an integer $t > 0$, we approximate (**) over R_n for some $n \in \beta$, i.e., we obtain

- (i) a complex $L_{\bullet,n}$ such that $L_{i,n}$ are all free except $T_{2r,n}$, $L_{\bullet,n} \otimes_{R_n} R_n/m_n^t \simeq L_\bullet \otimes_S S/m^t$ and $H_i(L_{\bullet,n}) \otimes_{R_n} R_n/m_n^t \simeq H_i(L_\bullet) \otimes_S S/m^t$ (Corollary 1 (3.5)).
- (ii) $\psi_\bullet: L_{\bullet,n} \rightarrow (F_{\bullet,n} \otimes_{R_n} G_{\bullet,n})_{\geq 1}$ such that if $P_{\bullet,n}$ denotes the mapping cone of ψ_\bullet , then $P_{\bullet,n} \otimes_{R_n} R_n/m_n^t \simeq P_\bullet \otimes_S S/m^t$ and $H_0(P_{\bullet,n}) = M_n \otimes_{R_n} N_n$. Now we make the following claim.

Claim. $P_{\bullet,n}$ is exact except at $i = 0$ and $H_0(P_{\bullet,n}) = M_n \otimes_{R_n} N_n$.

This claim proves our theorem, since exactness of mapping cone of ψ_\bullet implies $H_{i+1}(F_{\bullet,n} \otimes_{R_n} G_{\bullet,n}) \simeq H_i(L_{\bullet,n})$ for $i \geq 0$ and hence $\text{Tor}_i^{R_n}(M_n, N_n) \otimes_{R_n} R_n/m_n^t \simeq H_i(F_{\bullet,n} \otimes_{R_n} G_{\bullet,n}) \otimes_{R_n} R_n/m_n^t \simeq H_i(L_{\bullet,n}) \otimes_{R_n} R_n/m_n^t \simeq H_i(L_\bullet) \otimes_S S/m^t = \text{Tor}_i^S(M, N) \otimes_S S/m^t$, for $i \geq 0$.

Proof of Claim. Consider the diagram

$$\begin{array}{ccccccc}
0 \rhd T_{2r,n} & \longrightarrow & L_{2r-1,n} & \longrightarrow & \cdots \longrightarrow & L_{r-1,n} & \longrightarrow \cdots \longrightarrow L_{0,n} \longrightarrow 0 \\
& & \downarrow \psi_{2r-1} & & & \downarrow \psi_{r-1} & & \downarrow \psi_0 \\
0 \rhd (F_{\bullet,n} \otimes_{R_n} G_{\bullet,n})_{2r} & \rhd & \cdots \rhd & (F_{\bullet,n} \otimes_{R_n} G_{\bullet,n})_r & \rhd & \cdots \rhd & (F_{\bullet,n} \otimes_{R_n} G_{\bullet,n})_1 \rhd (F_{\bullet,n} \otimes_{R_n} G_{\bullet,n})_0 \rhd 0
\end{array}$$

Write $P_{i,n} = (F_{\bullet,n} \otimes_{R_n} G_{\bullet,n})_i \oplus L_{i-2,n}$. Then the mapping cone of ψ_\bullet can be written as

$$\begin{aligned}
P_{\bullet,n}: 0 \rightarrow P_{2r+2,n} \rightarrow P_{2r+1,n} \rightarrow P_{2r,n} \rightarrow \cdots \xrightarrow{\alpha_{r+1,n}} P_{r+1,n} \rightarrow P_{r,n} \rightarrow \cdots \\
\rightarrow P_{1,n} \rightarrow P_{0,n} \rightarrow 0.
\end{aligned}$$

We have the following

- (a) $P_{\bullet,n} \otimes_{R_n} R_n/m_n^t \simeq P_\bullet \otimes_S S/m^t$,
- (b) P_\bullet is exact except at $i = 0$, and
- (c) $P_{\bullet,n}$ is exact for $i > r + 1$, and $\ell(H_i(P_{\bullet,n})) < \infty$ for $i \leq r + 1$.

Write $G_{r+1,n} = \text{coker } \alpha_{r+1,n}$ and $G_\bullet = \text{coker } \alpha_{r+1}$. Because of Remark (3.4), Corollary 1 (3.5), and (i) and (ii) above, we have $\text{Tor}_i^{A_n}(G_{r+1,n}, A_n/m_n^t) \simeq \text{Tor}_i^{\hat{A}}(G_{r+1}, \hat{A}/\hat{m}^t)$ for $i \geq 0$. Hence by Remark 2 (3.5), we have

$$\text{depth}_{R_n} G_{r+1,n} = \text{depth}_S G_{r+1} = \text{depth } \ker \alpha_r = r = \text{depth } S > 0.$$

Hence $H_{r+1}(P_{\bullet,n}) = 0$ (since $\ell(H_{r+1}(P_{\bullet,n})) < \infty$ and $H_{r+1}(P_{\bullet,n}) \hookrightarrow G_{r+1,n}$). Repeating this argument a finite number of times, we obtain our claim. \square

We next prove our final result.

3.7. Theorem. *Let S be as above and let M and N be two finitely generated S -modules such that*

- (i) $\ell(M \otimes_S N) < \infty$,
- (ii) $\text{pd}_S M < \infty$, $\text{pd}_S N < \infty$, and
- (iii) $\dim M + \dim N \leq \dim S$.

Then \exists a local ring R of essentially finite type over k , a pair of finitely generated modules M' , N' over R such that

- (i) $\ell(M' \otimes_R N') < \infty$,
- (ii) $\text{pd}_R M' < \infty$, $\text{pd}_R N' < \infty$,
- (iii) $\dim M' + \dim N' \leq \dim R$ and $\chi^R(M', N') \geq 0$ according as $\chi^S(M, N) \geq 0$.

Proof. Choose $t \gg 0$ such that $m^t(M \otimes_S N) = 0$. By Theorem 3.6, with respect to this t , $\exists n \in \beta$, a local ring R_n and a pair of R_n -modules M_n, N_n such that $\text{Tor}_i^{R_n}(M_n, N_n) \otimes R_n/m_n^t \simeq \text{Tor}_i^S(M, N) \otimes S/m^t$ for $\forall i \geq 0$. Since $m^t(M \otimes_S N) = 0$, we have $m^t \text{Tor}_i^S(M, N) = 0$ and $m_n^t \text{Tor}_i^{R_n}(M_n, N_n) = 0 \forall i \geq 0$. Thus $\chi^{R_n}(M_n, N_n) \geq 0$, according as $\chi^S(M, N) \geq 0$. \square

Remark. The above proof works equally well in the mixed characteristics.

Proof of Theorem 3.1. Combining Theorem 3.7 and Proposition 3.2 we obtain our required proof. Note that for intersection multiplicity, one can always assume that B is complete. \square

Remark. Similar result holds good for the strong intersection conjecture of Peskine and Szpiro (i.e., if $\text{pd}_B M < \infty$ and $\ell(M \otimes_B N) < \infty$, then $\dim N \leq \text{grade } M$).

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